Motion in $\vec{E}$ & $\vec{B}$ fields

(1) Motion in constant, uniform $\vec{B}$ $(\vec{E}=0)$

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \qquad \frac{dU}{dt} = q\,\vec{v}\cdot\vec{E}$$

here $\vec{E}=0$, $U = \text{const} \Rightarrow \gamma = \text{const}$

Recall $\vec{p} = \gamma m \vec{v}$

so
$$\frac{d\vec{v}}{dt} = \vec{v}\times\frac{q}{\gamma m c}\vec{B} \equiv \vec{v}\times\vec{\omega}_B$$

$$\omega_B \equiv \frac{qB}{\gamma m c}$$ gyration or precession frequency

if $\vec{B}$ is along z direction

$$\frac{dv_x}{dt} = v_y\,\omega_B \qquad \frac{dv_y}{dt} = -v_x\,\omega_B \qquad \frac{dv_z}{dt} = 0$$

Will be convenient to have coordinate free form

$$\vec{v}(t) = v_{\parallel}\hat{\epsilon}_3 + \omega_B a\left(\hat{\epsilon}_1 - i\hat{\epsilon}_2\right)e^{-i\omega_B t}$$

$\hat{\epsilon}_3$ along $\vec{B}$ $\quad$ $\hat{\epsilon}_1, \hat{\epsilon}_2$ orthogonal unit vectors to $\vec{B}$

we really mean $\quad \vec{v}(t) = \mathrm{Re}\left\{\text{above expression}\right\}$

$$\vec{v}(t) = v_{\parallel}\hat{\epsilon}_3 + \omega_B a\left(\cos\omega_B t\,\hat{\epsilon}_1 - \sin\omega_B t\,\hat{\epsilon}_2\right)$$

for RH system $\quad \hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{\epsilon}_3$

$$\vec{x}(t) = \vec{X}_0 + v_{\parallel} t\,\hat{\epsilon}_3 + ia\left(\hat{\epsilon}_1 - i\hat{\epsilon}_2\right)e^{-i\omega_B t}$$

ie. $\quad \vec{x}(t) = \vec{X} + v_{\parallel} t\,\hat{\epsilon} + a\sin\omega t\,\hat{\epsilon} + a\cos\omega t\,\hat{\epsilon}$

$$a = \text{gyration radius} = \frac{v_\perp}{\omega_B} = \frac{\gamma m v_\perp c}{q B} = \frac{c p_\perp}{q B}$$

Numerically

$$\frac{p_\perp}{\text{MeV}/c} = 3.00 \times 10^{-4} \frac{B}{\text{Gauss}} \frac{a}{\text{cm}} \quad \text{for } |q| = e$$

in galaxy $B \simeq 3 \times 10^{-6}$ Gauss   in cluster 10^{-6} Gauss

Suppose $|\vec{E}| < |\vec{B}|$. find frame in which $\vec{E} = 0$

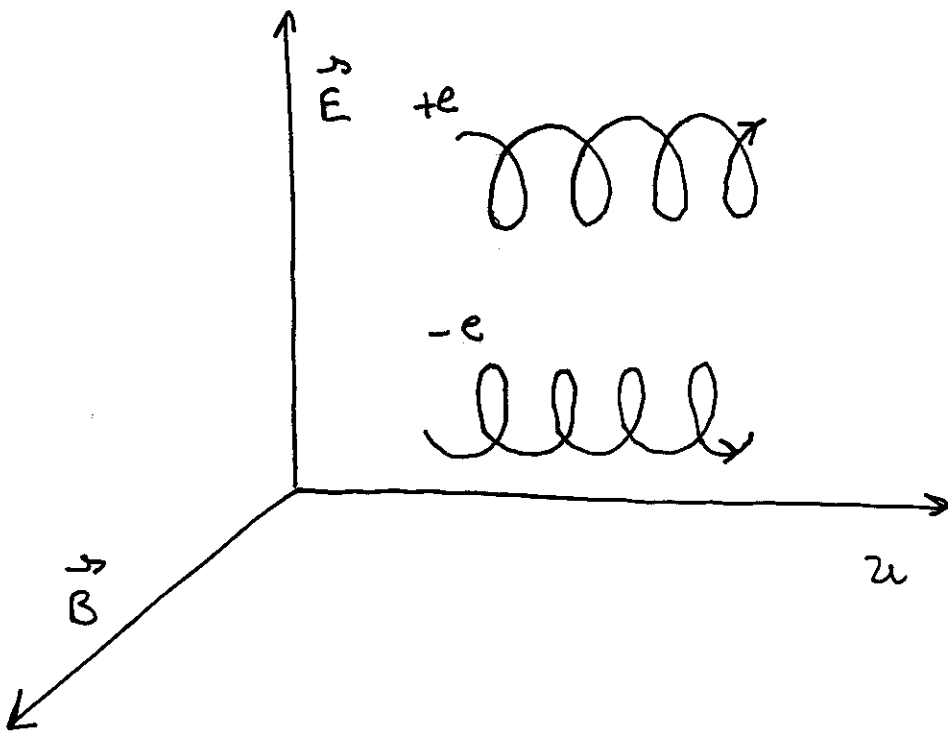
a) choose $\vec{u}$ = vel. of K' rel to K $\quad \vec{\beta} = \vec{u}/c$

b) $\vec{u}, \vec{E}, \vec{B}$ mutually perpendicular

c) $\vec{\beta}\cdot\vec{E} = 0$ $\quad$ want $\quad \vec{E} + \vec{\beta}\times\vec{B} = 0$

so $\quad \vec{B}' = \frac{1}{\gamma}\vec{B} = \left(1 - \frac{E^2}{B^2}\right)^{1/2} \vec{B}$

$$\vec{\beta} = \frac{E}{B}\hat{\beta} \qquad \gamma = \left(1 - \frac{E^2}{B^2}\right)^{-1/2}$$



Both +e + -e drift in same direction. + in plane containing $\vec{u}$ and $\vec{E}$

Case $|\vec{E}| > |\vec{B}|$ - transform to frame in which $\vec{B} = 0$ ...

Note $\vec{E}\cdot\vec{B}$ and $E^2 - B^2$ are Lorentz invariants

So if $\vec{E}\cdot\vec{B} = 0$ $(\vec{E}\perp\vec{B})$ + $|\vec{E}| < |\vec{B}|$

we can find a frame in which $\vec{E}$ vanishes

If $\vec{E}$ + $\vec{B}$ not $\perp$, this wont be possible.

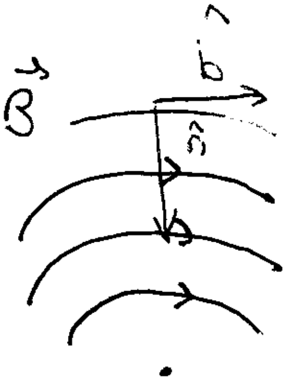
(3) Non-uniform Static fields

Assume distance over which fields change is large compared to gyration radius of particle.

Then we have spiralling along field lines
Gyration radius is set by local field strength
Center of gyration drifts slowly
Slow variations in gyration rate

Consider case where $\vec{\nabla} B \perp \vec{B}$

e.g. field of wire



particle $\underline{\underline{\text{not}}}$ moving along direction of $\vec{B}$!

$\vec{B} = B(r)\hat{\phi}$ so gradient is in $\hat{\rho}$ direction for $\hat{\phi}$ field.

Use Taylor expansion to describe fields.

(first guess what happens to particle $\underline{\underline{\text{not}}}$ moving along field lines)

$$\vec{B}(\vec{x}) = \vec{B}(\vec{x}_0) + (\vec{x} - \vec{x}_0) \cdot \vec{\nabla} \vec{B}(\vec{x}_0)$$

set $\vec{x}_0 = 0$, define $\vec{B}(\vec{x}_0) = \vec{B}_0$

$\vec{B} = B(\xi)\hat{b}$ $\qquad$ ξ coordinate along $\hat{n}$ - direction of gradient

$\vec{x} \cdot \hat{n} = \xi$

$$\vec{B}(\vec{x}) = \vec{B}_0 + (\vec{x} \cdot \hat{n})\left(\frac{\partial B}{\partial \xi}\right)_0 \hat{b} = \vec{B}_0\left(1 + (\hat{n} \cdot \vec{x})\frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0\right)$$

$$\vec{\omega}_B(\vec{x}) = \frac{e}{\gamma m c}\vec{B}(\vec{x}) = \vec{\omega}_0\left(1 + \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \hat{n}\cdot\vec{x}\right)$$

Now $$\frac{d\vec{v}_\perp}{dt} = \vec{v}_\perp \times \vec{\omega}_B$$ *assume no motion along field lines*

write $\vec{v}_\perp = \vec{v}_0 + \vec{v}_1$

$$\frac{d}{dt}(\vec{v}_0 + \vec{v}_1) = (\vec{v}_0 + \vec{v}_1) \times \vec{\omega}_0\left(1 + \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \hat{n}\cdot\vec{x}\right)$$

Recall $\quad \vec{v}_0 = \omega_0 a (\hat{\epsilon}_1 - i\hat{\epsilon}_2) e^{-i\omega_0 t}$

$$\vec{x}_0 = v_{\parallel} t \hat{\epsilon}_3 + ia(\hat{\epsilon}_1 - i\hat{\epsilon}_2) e^{-i\omega_0 t} \qquad \underline{X} = 0$$

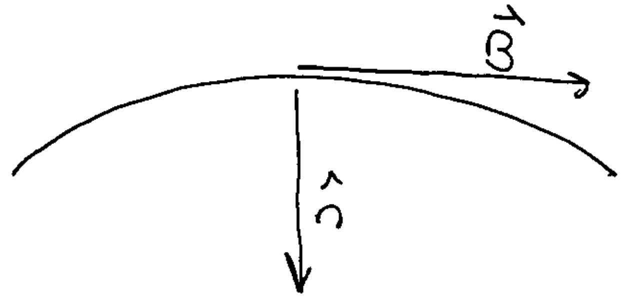
center of gyration radius

$$\omega_0 \times \epsilon_1 = \omega_0 \epsilon_2$$

gives $\quad \vec{v}_0 = -\vec{\omega}_0 \times \vec{x}_0$

$$\frac{d\vec{v}_1}{dt} = \left[\vec{v}_1 - \frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_0 \times \vec{x}_0 (\hat{n}\cdot\vec{x}_0) \right] \times \vec{\omega}_0$$

Quick and dirty



Particle orbits in plane containing $\hat{n}$ + $\perp$ to $\vec{B}$

$\hat{\epsilon}_1$ out of page

$$\vec{x}_0 = a(\sin\omega_B t\,\hat{\epsilon}_1 + \cos\omega_B t\,\hat{n})$$

$$\langle \vec{x}_0(\hat{n}\cdot\vec{x}_0)\rangle = a^2\langle \sin\omega_B t\,\cos\omega_B t\,\hat{\epsilon}_1 + \cos^2\omega_B t\,\hat{n}\rangle$$

$$= \frac{a^2}{2}\hat{n}$$

Gradient drift velocity is $\vec{v}_G = \frac{a^2}{2}\frac{1}{B_0}\left(\frac{\partial B}{\partial \xi}\right)_0 \vec{\omega}_B \times \hat{n}$

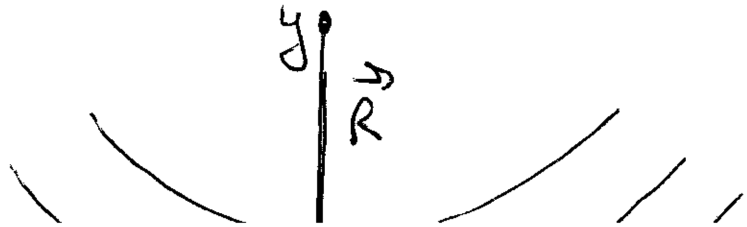
$$\vec{\omega}_B = \omega_B \frac{\vec{B}}{B}$$

$$\vec{\nabla}_\perp B = \hat{n}\frac{\partial B}{\partial \xi}$$

$$\boxed{\frac{\vec{v}_G}{a\omega_B} = \frac{a}{2B^2}\vec{B}\times\vec{\nabla}_\perp B}$$

positively + negatively charged particles drift in opposite directions

More generally, particle spirals along field lines



z out of page

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

$$\vec{v} \times \vec{\omega}_B = \omega_0 \frac{R}{\rho}\dot{\rho}\hat{z} - \omega_0 \frac{R}{\rho}\dot{z}\hat{\rho}$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

① $\ddot{\rho} - \rho\dot{\phi}^2 = -\omega_0 \frac{R}{\rho}\dot{z}$

② $\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = 0$

③ $\ddot{z} = \omega_0 \frac{R}{\rho}\dot{\rho}$

② conservation of ang. momentum $\rho^2\dot{\phi} = L = \text{constant}$

$$\dot{z} = R\omega_0 \ln \rho/R + v_0 \qquad v_0 \text{ integration constant}$$

$$\ddot{\rho} - \frac{L^2}{\rho^3} = -\omega_0^2 R^2 \frac{\ln \rho/R}{\rho} - \omega_0 v_0 \frac{R}{\rho}$$

Now we assume $\omega_0 R = v_\perp$

$$\gg \begin{cases} v_0 \\ L/R = v_\parallel \end{cases}$$

then
$$\frac{X_{eq}}{R} = \frac{L^2}{\omega_0^2 R^4} - \frac{v_0}{\omega_0 R}$$

$$= \frac{v_\parallel^2}{\omega_0^2 R^2} - \frac{v_0}{\omega_0 R}$$

$$= \frac{\langle x \rangle}{R}$$

$$\langle \dot{z} \rangle = v_0 + \omega_0 \langle x \rangle = \frac{v_\parallel^2}{\omega_0 R}$$

$$= v_c$$ curvature drift

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