Relativistic Electrodynamics

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Now write equations in terms of potentials

$$\vec{\nabla} \cdot \left(-\vec{\nabla} \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = -\nabla^2 \Phi - \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = 4\pi\rho$$

Can we do this?

Back to fields

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}$$

$$\partial^\alpha = \left(\frac{1}{c}\frac{\partial}{\ } , -\frac{\partial}{\ } , \frac{\partial}{\ } , \frac{\partial}{\ } \right) \qquad \partial$$

field equations

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial^\beta \underbrace{\cancel{\partial_\alpha A^\alpha}}_{\to 0 \text{ in L.G.}}$$

$$= \Box A^\beta = \frac{4\pi}{c} J^\beta$$

takes care of 2 "source" equations

Note $\partial_\beta \partial_\alpha F^{\alpha\beta} = 0 = \frac{4\pi}{c} \partial_\beta J^\beta$ ✓ Maxwell term! Charge cons.

$$F_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} F^{\gamma\delta}$$

$\vec{E} \to -\vec{E}$

$\vec{B} \to \vec{B}$

$$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ & 0 & -B_z & B_y \\ & & 0 & -B_x \\ & & & 0 \end{pmatrix}$$

for homogeneous equations, define

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \alpha = 0,\ \beta = 1,\ \gamma = 2,\ \delta = 3 \ + \text{even perm.} \\ -1 & \text{for odd permutations} \\ 0 & \text{if any two are same} \end{cases}$$

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$$

$$= \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ & 0 & E_z & -E_y \\ & & 0 & E_x \\ & & & 0 \end{pmatrix}$$

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0$$

$\beta = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

$\beta = 1,2,3 \Rightarrow \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

Lorentz equation

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

we'll take $\vec{p}$ to be space part of 4-momentum

$$\frac{d\vec{p}}{d\tau} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\frac{dt}{d\tau}$$

$$\frac{dt}{d\tau} = \gamma = \frac{u^0}{c} \qquad \frac{dt}{d\tau}\vec{v} = \gamma\vec{v} = \vec{u}$$

so $$\frac{d\vec{p}}{d\tau} = \frac{q}{c}\left(u^0 E^x + u^y B^z - u^z B^y\right)$$

$$= \frac{q}{c}\left(F^{10}u_0 + F^{12}u_2 + F^{13}u_3\right)$$

$$+ \cancel{F^{11}u_1}^{\,0}$$

$$\boxed{\frac{dp^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}u_\beta}$$

we get an extra equation

$$\frac{dp^0}{d\tau} = \frac{q}{c}F^{0\beta}u_\beta = \frac{q\gamma}{c}\vec{E}\cdot\vec{v}$$

$$\Rightarrow \quad \frac{dU}{dt} = q\vec{E}\cdot\vec{v} \qquad \text{work energy!}$$

$U = cp_0$ particle

Transformation of fields

Suppose we have $\vec{E}, \vec{B}$ in frame K + we want fields $\vec{E}', \vec{B}'$ in K'

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x'^{\beta}}{\partial x^{\gamma}} F^{\delta\gamma}$$

$$\frac{\partial x'^{\alpha}}{\partial x^{\delta}} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = A$$

for K' moves at velocity $\vec{v} = v\hat{x}$ relative to K

$$F' = A F \tilde{A}$$

$$E_1' = E_1 \qquad E_2' = \gamma(E_2 - \beta B_3) \qquad E_3' = \gamma(E_3 + \beta B_2)$$

$$B_1' = B_1 \qquad B_2' = \gamma(E_2 + \beta B_3) \qquad B_3' = \gamma(E_3 - \beta B_2)$$

or $$E_{\parallel}' = E_{\parallel} \qquad \vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B})$$

Now $\quad \vec{E}_{\parallel} = \vec{\beta}\,\dfrac{\vec{\beta}\cdot\vec{E}}{\beta^2} \qquad \vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp}$

so $\quad \vec{E}' = \vec{\beta}\,\dfrac{\vec{\beta}\cdot\vec{E}}{\beta^2} + \gamma\left(\vec{E}_{\perp} + \vec{\beta}\times\vec{B}\right)$

Field of a charge in constant motion

K' rest frame of charge $\qquad K$ rest frame of observer

Observer at $P \qquad x_1 = x_3 = 0 \quad x_2 = b$

or $\qquad x_1' = -vt', \; x_3' = 0, \; x_2' = b$

find fields in K' - simple Coulomb fields

$$r' = \left(x_1'^2 + x_2'^2 + x_3'^2\right)^{1/2} = \left((vt')^2 + b^2\right)^{1/2}$$

$$E_1 = E_1' \qquad E_2 = \gamma E_2'$$

so $$E_1 = -\frac{qvt'}{r'^3} \qquad E_2' = \frac{\gamma q b}{r'^3}$$

need t' in terms of t

so $\quad \vec{E} = \dfrac{q\gamma}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \left(-vt,\ b,\ 0\right)$

$\vec{r} = (-vt, b, 0)$ vector from particle to observer

$$\gamma^2 v^2 t^2 + b^2 = \gamma^2 r^2 + (1-\gamma^2) b^2$$

$$= \gamma^2 r^2 \left(1 - \frac{\gamma^2 - 1}{\gamma^2} \frac{b^2}{r^2}\right) \qquad \frac{1}{\gamma^2} = 1 - \beta^2$$

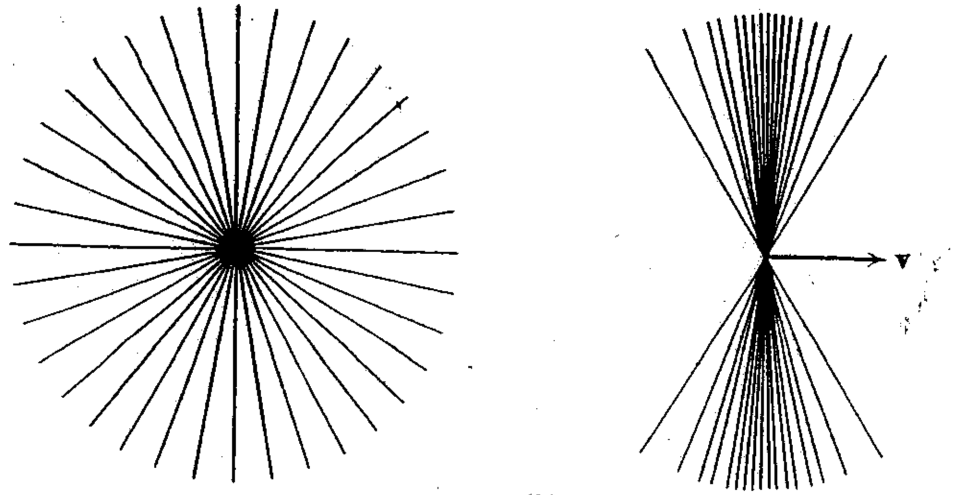
$$= \gamma^2 r^2 \left(1 - \beta^2 \sin^2 \psi\right) \qquad \sin\psi = b$$

so $\quad \vec{E} = \dfrac{q\vec{r}}{\gamma^2 r^3 (1 - \beta^2 \sin^2\psi)^{3/2}}$ $\quad$ points back to particle!

for $\psi = \pi/2$ (particle crosses $x=0$) $\quad \vec{E} = \dfrac{q}{b^2} \hat{y} \times \gamma$

increased by γ

$\psi = 0$ $\quad$ observer on x-axis $\quad \vec{E} = \dfrac{q}{(vt)^2} \hat{x} \ \dfrac{1}{\gamma^2}$



(b)

Fig. 11.9 Fields of a uniformly moving charged particle. (a) Fields at the observation point P in Fig. 11.8 as a function of time. (b) Lines of electric force for a particle at rest and in motion ($\gamma=3$).

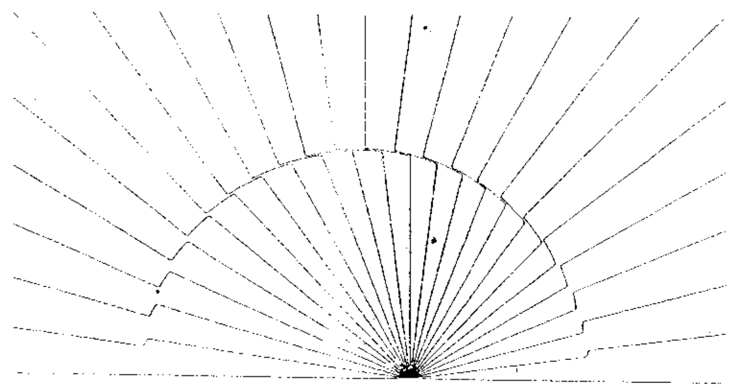


Fig. 11 Electric field lines of a

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