Euclidean geometry

follows from postulates

or consequence of assumption that space is described by metric

$$ds^2 = dx^2 + dy^2 + dz^2$$

distance between $(x, y, z)$ + $(x+dx, y+dy, z+dz)$ is $ds$.

Vectors directed

dot product

$$ds^2 = d\vec{x} \cdot d\vec{x}$$

$$= g_{ij}\, dx^i dx^j \qquad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

implied sum on $i$ & $j$

Note position of indices!

metric tensor

Call $dx^i$, $A^i$, ... covariant vector

Define $dx_i = g_{ij}\, dx^j$ contravariant vector or dual vector

$$ds^2 = dx \; dx^j$$

$ds^2$ is invariant under coordinate transformations e.g. $\quad x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{dx}{d\phi}d\phi$$

$$= \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta + r\sin\theta\sin\phi\, d\phi$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

$$g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$$

Euclidean space in spherical-polar coordinates

In general

$$dx^i = \frac{\partial x^i}{\partial x'^j}dx'^j$$

$$A^i = \frac{\partial x^i}{\partial x'^j}A'^j$$

transformation for <u>any</u> vector works same way prototypical dual vector

$$\nabla_i = \frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\frac{\partial}{\partial x^i} = \frac{\partial x^{j'}}{\partial x^i}\frac{\partial}{\partial x^{j'}}$$

$$A_i = \frac{\partial x^{j'}}{\partial x^i} A_j'$$

$$\vec{A}\cdot\vec{B} = A_i B^i = \frac{\partial x^{j'}}{\partial x^i} A_j' \frac{\partial x^i}{\partial x^{k'}} B^{k'}$$

$$= \delta^j_k A_j' B^{k'} = A_j' B^{j'}$$

special coordinate transformations leave form of metric unchanged - these are rotations + translations. Focus on rotations for rotation about z-axis

$$x = \cos\theta\, x' - \sin\theta\, y' \qquad y = \sin\theta\, x' + \cos\theta\, y'$$

$$\frac{\partial x^j}{\partial x^{i'}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \end{pmatrix}$$

We can also have 2 (or more) index tensors

$A_{ij}$ , $T^{i}{}_{kj}$ , ...

prototype $g_{ij}$ metric tensor

$$g_{ij} = \frac{\partial x^{k'}}{\partial x^{i}} \frac{\partial x^{l'}}{\partial x^{j}} g'_{kl}$$

$$\delta^{l}_{n}$$

$$ds^2 = g_{ij}\, dx^i dx^j = \frac{\partial x^{k'}}{\partial x^{i}} \frac{\partial x^{l'}}{\partial x^{j}} g'_{kl} \frac{\partial x^{i}}{\partial x^{m'}} \frac{\partial x^{j}}{\partial x^{n'}} dx^{m'} dx^{n'}$$

$$\delta^{k}_{m}$$

Special / General relativity

spacetime described by a <u>metric</u>

In special relativity, interval $ds$ defined by

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$= (dx^0)^2 - d\vec{x} \cdot d\vec{x}$$

$$= \eta_{\alpha\beta}\, dx^\alpha dx^\beta \qquad \eta_{\alpha\beta} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

<u>interval between events</u> $t, x, y, z$ + $t+dt, x+dx, \ldots$

If events are timelike separated (e.g. two events along ...

If two events are spacelike separated then

$$ds^2 = -dl^2 \qquad dl = \text{proper distance}$$

= distance between events measured in frame in which they occur at same time

$ds^2$ invariant under general coordinate transformations

e.g. transform to cylindrical coordinates

$$ds^2 = c^2dt^2 - dr^2 - r^2d\phi^2 - dz^2$$

then transform to rotating frame $\phi = \phi' - \omega t'$
$t = t'$

$$ds^2 = c^2dt^2 - dr^2 - r^2(d\phi - \omega dt)^2 - dz^2$$

I've dropped primes

<u>special</u> set of transformations leave form of metric unchanged

3 rotations + 3 Lorentz transformations (+ translations)

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$dx^\alpha$ is vector $\qquad \eta_{\alpha\beta}\, dx^\alpha = dx_\beta$ dual vector

$ds^2 = dx_\beta\, dx^\beta$ dot product of two vectors - a scalar

Coordinate transformations

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x^{\beta'}}\, dx^{\beta'}$$

e.g. $\quad x^0 = \gamma(x^{0'} + \beta x^{1'}) = \cosh\psi\, x^{0'} + \sinh\psi\, x^{1'}$

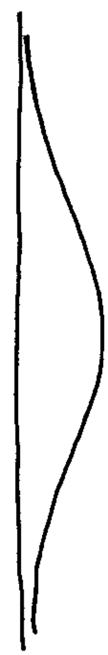
$\quad x^1 = \gamma(x^{1'} + \beta x^{0'}) = \sinh\psi\, x^{0'} + \cosh\psi\, x^{1'}$

$$\frac{\partial x^\alpha}{\partial x^{\beta'}} = \begin{pmatrix} \cosh\psi & \sinh\psi & & \\ \sinh\psi & \cosh\psi & & \\ & & & \end{pmatrix}$$

$$ds^2 = +(dx^{0\prime})^2 - (dx^{1\prime})^2 - (dx^{2\prime})^2 - (dx^{3\prime})^2$$

form is unchanged under special transformations.

Twin paradox



accelerating twin

$$\Delta\tau = \int dt\,(1-\beta^2)^{1/2}$$

$$< \Delta\tau_{\text{non-accel.}}$$

If we work in rest frame of accelerating twin then

Vectors + dual vectors

$$A^\alpha = (A^0, \vec{A})$$

$$A_\alpha = g_{\alpha\beta} A^\beta = (A^0, -\vec{A})$$

$$\underset{\sim}{A} \cdot \underset{\sim}{B} = \eta_{\alpha\beta} A^\alpha B^\beta$$

$$= A_\alpha B^\alpha = A^0 B^0 - \vec{A} \cdot \vec{B}$$

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left( \frac{\partial}{\partial x^0}, \vec{\nabla} \right) = \left( \frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right)$$

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left( \frac{\partial}{\partial x^0}, -\vec{\nabla} \right) = \left( \frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right)$$

$$\partial^\alpha A_\alpha = \partial A^0 \ldots \vec{\nabla} \cdot \vec{A}$$

Define 4-vector $J^\alpha = (c\rho, \vec{J})$

$$\partial_\alpha J^\alpha = \frac{1}{c}\frac{\partial}{\partial t} c\rho + \vec{\nabla}\cdot\vec{J} = 0 \quad ✓$$

Also, let $A^\alpha = \partial^\alpha f$ $\quad \partial_\alpha A^\alpha = \partial_\alpha \partial^\alpha f = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f$ wave operator

D'alembertian

$\Box f$

4 - velocity

Let $dx^\alpha$ by element along worldline

V 13

[redacted]

$$d\tau^2 = \frac{1}{c^2} ds^2 = \frac{1}{c^2} g_{\alpha\beta}\, dx^\alpha dx^\beta$$

or $\quad 1 = \frac{1}{c^2} g_{\alpha\beta} \frac{dx^\alpha}{\,} \frac{dx^\beta}{\,}$ [illegible]