Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$ Coulomb's law

$$\frac{1}{8\pi}\left(E^2 + B^2\right) = \text{energy density}$$

$$1\ \text{erg}/\text{cm}^3 = \left(\text{statcoulomb}/\text{cm}^2\right)^2$$

$$= \text{dyne}/\text{cm}^2$$

$$= \text{gauss}^2$$

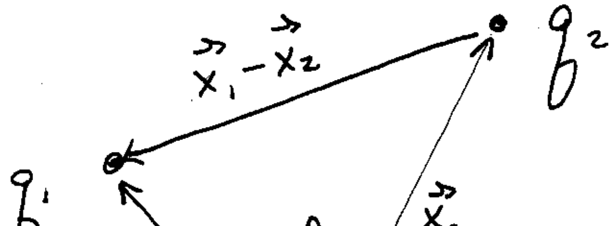
Ohm's law $\vec{J} = \sigma \vec{E}$ $\qquad [\sigma] = s^{-1}$

Electrostatics $\quad \vec{E}, \rho$ time independent

$\vec{J}, \vec{B} = 0$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \times \vec{E} = 0$$

Begin with Coulomb force law



$$\vec{F}_{\text{on 1 due}} = q_1 q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

for system of charges

$$\vec{E}(\vec{x}) = \sum_i q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|}$$

$$q_i \rightarrow \rho(\vec{x}\,')\, d^3x'$$

$$\vec{E}(\vec{x}) = \int \rho(\vec{x}\,') \; \vec{x} - \vec{x}\,' \; d^3$$

so $\quad \nabla_x |\vec{x} - \vec{x}'| = \dfrac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}$

$$\vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} = -\frac{1}{2}\left(\quad\right)^{-3/2} \left(2\vec{x} - 2\vec{x}'\right) = -\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \quad \checkmark$$

So $\quad \vec{E} = -\vec{\nabla}\Phi \qquad \Phi = \displaystyle\int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$

Note $\quad \vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{\nabla}\Phi = 0$

<u>Alternative</u>

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \times \vec{E} = 0$$

second equation implies $\vec{E} = -\vec{\nabla}\Phi$

then $\quad \nabla^2 \Phi = -4\pi\rho$

Solve using Green's functions

first solve for δ-fn source

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta^3(\vec{x}-\vec{x}')$$

then $$\Phi(\vec{x}) = \int d^3x'\, G(\vec{x},\vec{x}')\,\rho(\vec{x}')$$

we now show

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$$

i.e. $$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta(\vec{x}-\vec{x}')$$

let $\vec{r} = \vec{x}-\vec{x}'$

$$\nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \left(\frac{1}{r}\right) = 0 \quad \text{except where } r=0$$

Integrate $\nabla^2 \frac{1}{r}$ over ball of radius $r = R$ centered on origin

$$\int \nabla^2 \frac{1}{r} \, d^3 r = \int \left(\vec{\nabla} \cdot \vec{\nabla} \frac{1}{r} \right) d^3 r$$

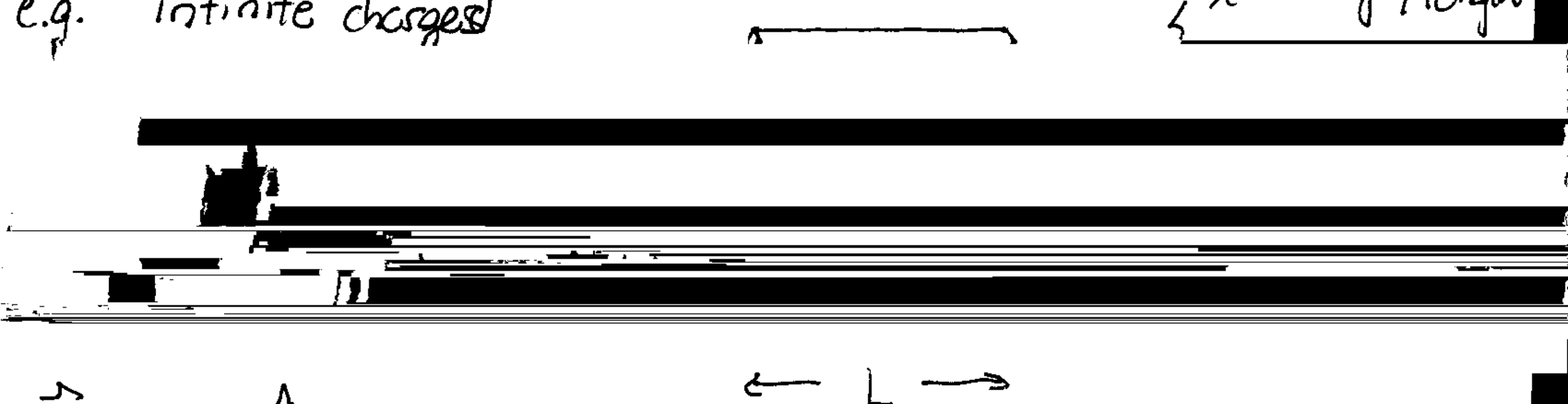
$$= \oint \vec{\nabla} \frac{1}{r} \cdot \hat{n} \, da = \oint \left(-\frac{1}{R^2} \right) R^2 d\Omega$$

$$= -4\pi$$

Gauss's law

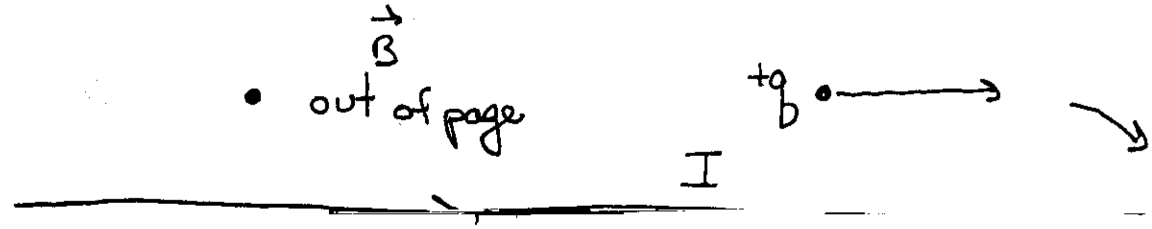
$$\int_V \vec{\nabla} \cdot \vec{E} \, d^3 x = \oint_S \vec{E} \cdot \hat{n} \, da = \int_V d^3 x \, 4\pi \rho$$

e.g. infinite charged $\qquad \lambda = $ charge/length



$$\vec{E} = E_s \hat{s}$$

Magnetic force law $\vec{F} = q\,\frac{\vec{v}}{c} \times \vec{B}$



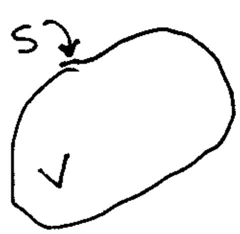
Magnetostatics $\vec{J}, \vec{B}$ time-independent

In general, charge is conserved

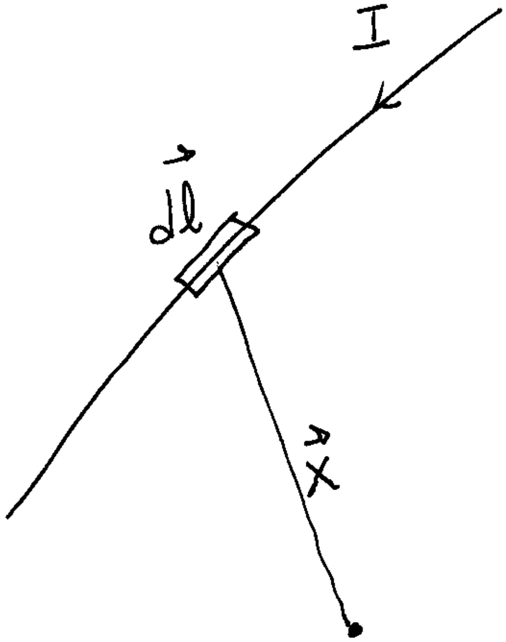
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad \text{continuity equation}$$

$$\int_V d^3x \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) = \overbrace{\frac{d}{dt} Q}^{\text{change of charge in } V} + \overbrace{\oint_S da\, \hat{n} \cdot \vec{J}}^{\text{flux out of } V \text{ thru } S} = 0$$

$$= 0$$



in magnetostatics, $\vec{\nabla} \cdot \vec{J} = 0$



$$d\vec{B} = \frac{I}{c}\,\frac{d\vec{l}\times\vec{x}}{|\vec{x}|^3}$$

but only makes sense for closed loop $(\vec{\nabla}\cdot\vec{J}=0)$

$[I]$ = statcoulombs/s
= statamps

$[\vec{J}]$ = statcoulombs/s/cm^2
= statamps/cm^2

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \left(\frac{1}{c} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x' \right)$$

$$= \vec{\nabla} \times \vec{A} \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0 \text{ automatically}$$

$$\vec{A}(\vec{x}) = \frac{1}{c} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x' + \vec{\nabla} \psi$$

gradient of scalar
gauge freedom